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Stochastic particle acceleration at shocks in the presence of braided magnetic fields

J.G. Kirk[1], **P. Duffy**[1], **Y.A. Gallant**[2]

[1] Max-Planck-Institut für Kernphysik, Postfach 10 39 80, D-69029 Heidelberg, Germany
[2] Sterrenkundig Instituut Utrecht, Postbus 80000, Utrecht 3508 TA, Netherlands

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Abstract. The theory of diffusive acceleration of energetic particles at shock fronts assumes charged particles undergo spatial diffusion in a uniform magnetic field. If, however, the magnetic field is not uniform, but has a stochastic or braided structure, the transport of charged particles across the average direction of the field is more complicated. Assuming quasi-linear behaviour of the field lines, the particles undergo sub-diffusion on short time scales. We derive the propagator for such motion, which differs from the Gaussian form relevant for diffusion, and apply it to a configuration with a plane shock front whose normal is perpendicular to the average field direction. Expressions are given for the acceleration time as a function of the diffusion coefficient of the wandering magnetic field lines and the spatial diffusion coefficient of the charged particles parallel to the local field. In addition we calculate the spatial dependence of the particle density in both the upstream and downstream plasmas. In contrast to the diffusive case, the density of particles at the shock front is lower than it is far downstream. This is a consequence of the partial trapping of particles by structures in the magnetic field. As a result, the spectrum of accelerated particles is a power-law in momentum which is steeper than in the diffusive case. For a phase-space density $f \propto p^{-s}$, we find $s = s_{\rm diff}[1 + 1/(2\rho_{\rm c})]$, where $\rho_{\rm c}$ is the compression ratio of the shock front and $s_{\rm diff}$ is the standard result of diffusive acceleration: $s_{\rm diff} = 3\rho_{\rm c}/(\rho_{\rm c} - 1)$. A strong shock in a monatomic ideal gas yields a spectrum of $s = 4.5$. In the case of electrons, this corresponds to a radio synchrotron spectral index of $\alpha = 0.75$.

Key words: acceleration of particles – diffusion – plasmas – shock waves – (ISM:) cosmic rays – ISM: supernova remnants

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Send offprint requests to: J.G. Kirk

1. Introduction

Diffusive particle acceleration at shock fronts has been advanced as an explanation for many astrophysical phenomena involving non thermal particles (for a review see Blandford & Eichler 1987). A key to the success of this theory is its simplicity: particles are assumed to diffuse in space upstream and downstream of a shock front, which is treated as a simple discontinuity in the velocity of the scattering centres. The well-known result is a power-law spectrum of accelerated particles $f(p) \propto p^{-s_{\rm diff}}$ with an index which is independent of the diffusion coefficient, being determined solely by the compression ratio $\rho_{\rm c}$ of the shock front: $s_{\rm diff} = 3\rho_{\rm c}/(\rho_{\rm c} - 1)$.

In a recent *Letter* (Duffy et al. 1995) it was pointed out that the transport of particles is more complicated in a 'braided' magnetic field, i.e., one which has a random component causing field lines to wander around when projected onto the plane normal to their average direction. In particular, it was shown that there may exist a regime of sub-diffusive transport, in which the average square displacement of a particle from its origin grows with the square-root of elapsed time. This differs sharply from the linear growth expected in the case of ordinary diffusion and reflects the restraining influence or trapping (in a stochastic sense) of charged particles by the magnetic field. At shock fronts where particles must be transported across the direction of the mean magnetic field in order to be accelerated, this leads to a modification of the acceleration rate.

In Sect. 2 we present a detailed discussion of the transport of particles in a braided field, pointing out the diffusive and sub-diffusive regimes. There are two distinct physical effects which come into play here: 'microscopic' diffusion i.e., that which would pertain in a uniform magnetic field as a result of fluctuations with a length scale approximately equal to the gyro radius of the particle concerned, and 'macroscopic' diffusion of the field lines themselves, which results from fluctuations of much longer length scale. The importance of braiding, or 'field line

wandering' as it is called in the astrophysical literature (Getmantsev 1963, Jokipii & Parker 1969, Jokipii 1973) can be quantified in terms of a single parameter Λ related to the correlation lengths of the macroscopic magnetic field fluctuations and to the relative strength of the macroscopic and microscopic fluctuations. In the case of plasmas in which braiding is important, a further distinction must be made concerning time scales. For long times, the transport is diffusive, and the spatial diffusion coefficient depends on a combination of the microscopic and macroscopic diffusivities. On short time scales, the transport is sub-diffusive. In this case, we present the analytic form of the particle propagator, pointing out that it tends to confine or trap the particles in a stochastic sense and is thus non-Markovian.

These ideas are applied in Sect. 3 to find the distribution of energetic particles accelerated in the vicinity of a shock front whose normal is perpendicular to the average direction of the field. At such shocks, the confining property of sub-diffusion has a strong influence on the acceleration process, since this is closely connected with the ability of particles to perform multiple crossings and recrossings of the shock front before being advected away with the downstream plasma. Assuming the angular distribution of the particles is kept close to isotropy by scattering off the microscopic fluctuations (an assumption which we discuss in Sect. 3), we compute the spectrum and the spatial dependence of the density of accelerated particles, using the exact form of the sub-diffusive propagator. Sub-diffusion manifests itself in a steeper spectral index than that given by the standard formula for diffusive acceleration. It also leads to a gradient of the particle density in the downstream plasma – a quantity which is strictly uniform in the diffusive case. The transition between sub-diffusive behaviour on short time scales and diffusive behaviour on longer ones can be investigated by constructing a propagator which switches from one form to the other at a certain point in time, which we call the decorrelation time. In Sect. 3 results are found using such a propagator, and the value of the decorrelation time for which particle acceleration is strongly affected by sub-diffusion is deduced.

Finally, in Sect. 4, we briefly discuss a few of the implications and limitations of our results. In particular, we point out that in synchrotron sources in which field line wandering and acceleration of electrons at shock fronts is important, systematically steeper spectra should be observed from locations at which the shock is predominantly perpendicular.

2. Cross-field transport

Our approach to the transport process involves separating it into two parts (Chuvilgin & Ptuskin 1993). The first of these consists of macroscopic magnetic fluctuations of length scale large compared to the gyro radius of the particle concerned, which we characterise by a relative amplitude $b \equiv \langle|\delta B|\rangle \,/\, \langle B\rangle$. As described in Duffy et al. (1995) we assume these fluctuations lead to a quasi-linear type diffusion or wandering of the field lines in the plane perpendicular to the direction of the average field. This is described by a magnetic diffusivity $D_{\rm M}$, which, in terms of cartesian coordinates with the z-axis along the average field, is defined by:

$$\frac{\left\langle \Delta x^2 \right\rangle}{2s} \;=\; D_{\rm M} \ , \tag{1}$$

where Δx is the change in the x co-ordinate upon travelling a distance s along the field line. If the turbulence is characterised by correlation lengths $\lambda_\parallel$ and $\lambda_\perp$ along and across the average field, then assuming quasi-linear behaviour and adopting the normalisation of Kadomtsev & Pogutse (1979), we have

$$D_{\rm M} \;=\; \frac{b^2\lambda_\parallel}{4} \ . \tag{2}$$

(Note that different normalisations are used by Achterberg & Ball (1994) and Isichenko (1991a)).

The second component of the transport concerns length scales comparable to that of the gyro radius. We assume that any anisotropy in the distribution function results in the rapid growth of magnetic fluctuations (Alfvén waves) which, in their turn, scatter the particles so as to remove the anisotropy. This process we model in terms of two spatial diffusion coefficients $\kappa_\parallel$ and $\kappa_\perp$ responsible for transport along and across the local field direction, respectively. In keeping with other treatments (e.g., Achterberg & Ball 1994, Jokipii 1987) we parameterise these coefficients in terms of the gyro-Bohm diffusion coefficient $\kappa_{\rm B} \equiv \gamma v^2 mc/(3eB)$, for a particle of mass m, charge e moving at speed v, with Lorentz factor $\gamma = (1-v^2/c^2)^{-1/2}$ in a magnetic field B:

$$\begin{aligned} \kappa_\parallel \;&=\; \frac{\kappa_{\rm B}}{\epsilon} \\ \kappa_\perp \;&=\; \frac{\epsilon\kappa_{\rm B}}{(1+\epsilon)} \ . \end{aligned} \tag{3}$$

Here $\epsilon \leq 1$ is the ratio of the energy density in microscopic fluctuations to that in the average magnetic field. In the picture in which the scattering is modelled by the "τ" operator (Chuvilgin & Ptuskin 1993), ϵ is the ratio of the collision rate to the gyro-frequency.

Thus we have on the one hand microscopic fluctuations of relative amplitude $\sqrt{\epsilon}$ which are responsible for the diffusion of particles along and across the local field and, on the other, macroscopic fluctuations of relative amplitude b which are responsible for diffusion of the field lines. We shall assume $\epsilon \ll 1$, which means that on the microscopic scale, particles are closely tied to field lines along which they diffuse. Transport across the local field direction is therefore severely restricted on the microscopic scale.

The important physical process in such a picture has been described by Rechester and Rosenbluth (1978): because of the exponential divergence of neighbouring field lines, a particle's orbit will take it out of regions of correlated magnetic field. This can happen either because the particle diffuses a small distance across the field by scattering off the microscopic fluctuations, or, in the case of very small ϵ, because the finite size of its gyro orbit causes it to encounter uncorrelated field lines (Isichenko 1991a). The latter case we term "ballistic" propagation. It occurs if the particle decorrelates from the field before having a chance to diffuse microscopically, which translates into the condition

$$\frac{b^2}{\epsilon} > \frac{\lambda_\perp^2}{\lambda_\parallel r_\mathrm{g}} \log\left(\frac{\lambda_\perp}{r_\mathrm{g}}\right) \tag{4}$$

(Duffy et al. 1995).

Ballistic propagation has been considered by Achterberg & Ball (1994) in connection with the radio emission of supernovae. The particles propagate diffusively, with an effective spatial diffusion coefficient across the field given by

$$\kappa_\perp^{\mathrm{ballistic}} = vD_\mathrm{M} \quad , \tag{5}$$

and the standard theory of diffusive acceleration at shocks applies. However, in this paper we will assume the self-excited microscopic turbulence is sufficiently strong to enable particles to scatter before they decorrelate from the magnetic field. In this case, one can define a dimensionless parameter

$$\Lambda = \frac{b^2\lambda_\parallel}{\sqrt{2}\epsilon\lambda_\perp} \tag{6}$$

such that for $\Lambda \lesssim 1$ the macroscopic braiding of the field is irrelevant, whereas it dominates cross-field transport for $\Lambda \gg 1$. If braiding is irrelevant, there is no anomalous transport regime, and no modification of the diffusive acceleration picture, provided the distribution can still be considered isotropic (cf. Achterberg & Ball 1994).

The most interesting parameter regime is that in which both braiding and microscopic scattering are important, which occurs for the parameter range

$$\frac{\lambda_\perp}{\sqrt{2}r_\mathrm{g}} \log\left(\frac{\lambda_\perp}{r_\mathrm{g}}\right) > \Lambda \gg 1 \quad . \tag{7}$$

Two kinds of propagation are then possible: for times less than that needed to decorrelate from the field t_d, the particles undergo sub-diffusion, which is a combination of diffusion along a fixed field line, which itself diffuses. The defining characteristic of sub-diffusion is that the mean square deviation of a particle in the x direction from its position at time $t = 0$ is not proportional to t as in ordinary diffusion, but rather to $\sqrt{t}$.

The propagator $P_\mathrm{sub}(x,t)$, which is the probability of finding a particle in the interval $(x,x+\mathrm{d}x)$ at time t, given that it was at the origin $x = 0$ at $t = 0$, is found by simply folding the two gaussian propagators appropriate to diffusion of a particle along the field (i.e., in s) and of the field in x:

$$P_\mathrm{sub}(x,t) = \frac{1}{4\pi\sqrt{D_\mathrm{M}\kappa_\parallel t}} \int_{-\infty}^{+\infty} \mathrm{d}s \frac{1}{\sqrt{|s|}} \exp\left(-\frac{x^2}{4D_\mathrm{M}|s|} - \frac{s^2}{4\kappa_\parallel t}\right) \quad . \tag{8}$$

(Rax & White 1992, Duffy et al. 1995). It is straightforward to confirm that the mean square value of x increases as $t^{1/2}$ with this propagator. Equation (8) describes the motion of an injected particle for times shorter than a decorrelation time, i.e.,

$$t < t_\mathrm{d} = \frac{(\lambda_\perp \log\Lambda)^2}{2\kappa_\parallel} \quad . \tag{9}$$

Subsequently, the particle decorrelates and undergoes "compound diffusion", which is the collisional transport regime discussed by Rechester & Rosenbluth (1978). Here the propagation is diffusive in character, with an effective diffusion coefficient given by a combination of macro- and microscopic effects:

$$\kappa_\mathrm{comp} \approx \kappa_\perp \left(1 + \frac{\Lambda^2}{\log\Lambda}\right) \tag{10}$$

The associated gaussian propagator is

$$P_\mathrm{comp}(x,t) = \frac{1}{\sqrt{4\pi\kappa_\mathrm{comp}t}} \exp\left[-\frac{x^2}{4\kappa_\mathrm{comp}t}\right] \quad . \tag{11}$$

Although we have derived the sub-diffusive propagator on the basis of quasi-linear macroscopic turbulence of the magnetic field, it is a phenomenon which probably has a much wider importance. This is demonstrated, for example, by the numerical work of Rax & White (1992), where sub-diffusion is found using a simple Taylor-Chirikov mapping to model the macroscopic field turbulence. Of course, the formulae we quote for quantities such as the decorrelation time (Eq. 9) are not generally applicable to other models of turbulence. Furthermore, they apply only in the absence of significant particle drifts and only if the magnetic field can be considered static. Both drifts and time-dependent fields may in fact be responsible for decorrelating the particle from a field line (Isichenko 1991b) before this is achieved by either diffusion across the field or by the finite size of the gyro orbit. Nevertheless, even though it does not appear possible to model the macroscopic turbulence in an astrophysical source in such detail, we may nevertheless expect the qualitative picture of a sub-diffusive regime on short time scales followed by a diffusive one on longer time scales to be a generic feature.

3. Acceleration at quasi-perpendicular shocks

In a uniform magnetic field, the microscopic fluctuations enable particles to diffuse in space. From the theory of diffusive acceleration at parallel shocks, it is well-known that the particle distribution at the shock front is almost isotropic, provided that the speed of the shock is small compared to the speed of the individual particles. Furthermore, if the magnetic field is not directed along the shock normal, but is oblique, then it has been shown that in the absence of cross-field transport the distribution is almost isotropic, given that the speed of the point of intersection of the magnetic field and the shock front is small compared to the particle speed. If the magnetic field makes an angle $\phi_{\rm up}$ with the shock normal as measured in the rest frame of the upstream plasma, this can be expressed as $\cos\phi_{\rm up} \gg u_{\rm s}/v$, where $u_{\rm s}$ is the speed of advance of the shock front into the upstream plasma and v is the particle speed (Kirk & Heavens 1989). For a slowly moving shock front, and for relativistic particles, this condition is fulfilled unless the magnetic field lies almost exactly in the plane of the shock. In a braided field, such points are relatively rare. A supernova remnant, for example, has a shock front moving at a speed of roughly $3000\,{\rm km\,s^{-1}}$, so that for a relativistic electron $u_{\rm s}/v \approx 1/100$ and the distribution remains close to isotropy unless the field is aligned to within about 1° of the shock plane. In the following, we assume that in a braided field, the microscopic fluctuations are able to keep the distribution close to isotropy at all points on the shock front and so neglect the effect of the small patches where the magnetic field is almost tangential to the shock surface.

3.1. Sub-diffusion

Several properties of the accelerated particle distribution can be found solely in terms of the single particle propagator $P(x-x',t-t')$. In particular, given a source function $Q(x,t)$ of injected particles the spatial density distribution $n(x,t)$ is

$$n(x,t) = \int_{-\infty}^{+\infty} {\rm d}x' \int_{-\infty}^{t} {\rm d}t' P(x-x',t-t')Q(x',t') \ . \quad (12)$$

Consider the case where particles are injected at a constant rate Q_0 for $t > t_0$ on a plane boundary initially at $x = 0$ which moves at constant speed u in the positive x–direction; i.e. $Q(x,t) = Q_0\delta(x-ut)H(t-t_0)$. The time asymptotic steady state distribution, which is found by allowing $t_0 \rightarrow -\infty$, is

$$n(y) \ = \ Q_0 \int_0^{\infty} {\rm d}t P(y+ut,t) \quad (13)$$

where $y = x - ut$ is the distance from the plane boundary. Far downstream of the boundary, $y \rightarrow -\infty$, we have $n(y) \rightarrow Q_0/u$. This can be seen by observing that far away from the boundary, the distribution should relax to a homogeneous state. Seen from the rest frame of the boundary, the flux of particles across any plane y = constant is just $un(y)$, and this must equal the rate of injection at $y = 0$. Alternatively, one may note that for a physically realistic propagator it is possible at any finite time t to choose spatial boundaries beyond which there are a negligible number of particles. Denoting these boundaries by $\pm x_0$, we can write

$$\int_{-x_0}^{x_0} {\rm d}x P(x,t) = 1; \quad P(x,t) = 0 \text{ for } |x| > x_0 \quad (14)$$

Then, substituting $x' = x/\alpha$ we have

$$\alpha \int_{-x_0/\alpha}^{x_0/\alpha} {\rm d}x' P(\alpha x',t) = 1 \quad (15)$$

or, equivalently, $P(\alpha x,t) \rightarrow \alpha^{-1}\delta(x)$, as $\alpha \rightarrow \infty$. This property, together with Eq. (13), implies $n(\infty) = 0$ (far upstream) and $n(-\infty) = Q_0/u$ (far downstream).

On the boundary, however, the particle density depends on the precise form of the propagator. Evaluating the integral in Eq. (13) with the sub-diffusive propagator Eq. (8), one finds $n(0) = 2Q_0/3U$, so that the ratio of the density far downstream to that at the boundary is

$$\frac{n(-\infty)}{n(0)} \ = \ \frac{3}{2} \quad (16)$$

In sub-diffusion, therefore, there is a spatial gradient downstream, which is absent in the case of diffusion. The scale length associated with this gradient can be obtained from the sub-diffusive propagator and is $D_{\rm M}^{2/3}\kappa_\parallel^{1/3}/U^{1/3}$.

As in the case of diffusive acceleration, these properties of the "*sub-diffusion-advection*" problem allow us to find the steady state phase space density $f_0(p)$ [$= n_0(p)/(4\pi p^2)$] of particles accelerated at a shock, provided we may assume that the shock itself does not affect the spatial transport (see Sect. 4). Given that particles suffer no energy losses during propagation and are accelerated only on crossing the shock front, conservation of particle number in momentum space requires the advected flux of particles with momentum p far downstream of the shock to equal the flux of particles across this momentum level at the shock (assuming no particles are advected in from far upstream). For an almost isotropic distribution, this flux is simply related to the momentum derivative at the shock (e.g., Drury 1987, Kirk et al. 1994). Writing the distribution function far downstream as $f_2(p)$, and denoting by u_1 and u_2 the upstream and downstream flow speeds in the shock one finds

$$\frac{\rm d}{{\rm d}p}\left(\frac{4\pi p^3}{3}(u_1-u_2)f_0(p)\right) + 4\pi p^2 u_2 f_2(p) \ = \ 0 \ , \quad (17)$$

giving a power law solution $f_{0,2} \propto p^{-s}$ with

$$s \ = \ 3\left[1 + \frac{f_2}{f_0(\rho_{\rm c}-1)}\right] \quad (18)$$

In the case of sub-diffusive propagation, $f_2/f_0 = n(-\infty)/n(0) = 3/2$, so that

$$s \;=\; \frac{3\rho_{\rm c}}{\rho_{\rm c}-1}\left(1+\frac{1}{2\rho_{\rm c}}\right) \tag{19}$$

For a strong shock, $\rho_{\rm c}=4$, Eq. (19) gives $s=4.5$, which is steeper than the spectrum produced by diffusive acceleration. Physically, this arises because particles are tied to individual field lines and are more effectively advected away from the shock than in the diffusive case.

This derivation may be re-cast slightly in terms of an *average* probability per cycle, $P_{\rm esc}$, for a particle to escape once it enters the downstream region (Bell 1978). In the steady state the ratio of the far downstream flux, $n(-\infty)|u_2|$, to the almost isotropic flux of particles of speed v crossing the shock, $n(0)v/4$, leads to $P_{\rm esc} = [n(-\infty)/n(0)](4|u_2|/v)$. Diffusion, for which the downstream gradient vanishes, leads to $P_{\rm esc}=4|u_2|/v$, whereas sub-diffusion gives $P_{\rm esc}=6|u_2|/v$. Thus, particles which are sub-diffusing have a higher probability of being advected away from the shock than those which are diffusing. Assuming isotropy, the mean increase of a particle's momentum on crossing and recrossing the shock is $\langle\Delta p\rangle = 4|u_1-u_2|p/3v$ from which the power law index of the phase space distribution can be calculated: $s=3+pP_{\rm esc}/\langle\Delta p\rangle$. For sub-diffusion, $P_{\rm esc}=6|u_2|/v$, we obtain Eq. (19).

It is straightforward to generalise the sub-diffusive result to any propagator of the form

$$P(x,t)=\frac{1}{t^{\beta/2}}\Phi\left(\frac{x}{t^{\beta/2}}\right) \tag{20}$$

which corresponds to transport with $\left\langle\Delta x^2\right\rangle \propto t^{\beta}$.[1] Using Eq. (13), the normalisation property $\int_{-\infty}^{\infty}{\rm d}x\Phi(x)=1$, and assuming $\Phi(x)$ is an even function of x, one finds $n(-\infty)=n(0)(2-\beta)$, leading to a spectral index

$$s \;=\; \frac{3\rho_{\rm c}}{\rho_{\rm c}-1}\left(1+\frac{1-\beta}{\rho_{\rm c}}\right)\quad . \tag{21}$$

3.2. Compound diffusion

So far we have considered only cases in which it is a good approximation to use a single form of the propagator to describe the particle transport. However, we have argued that sub-diffusion applies only for times less than $t_{\rm d}$, which is that needed for a particle to decorrelate from its original field line. Especially when calculating the stationary density profile, it is obvious that particles injected long ago should really be described by the appropriate diffusive propagator Eq. (11). There are several ways in which a propagator could be constructed with such properties, and one could certainly envisage using these in a numerical simulation. However, in order keep an analytically tractable form, we have been compelled to use a crude approximation. This consists of artificially changing the behaviour of each particle at a time $t_{\rm d}$ after it has been injected. Sub-diffusion is assumed at $t<t_{\rm d}$ and diffusion for all $t>t_{\rm d}$: The particle propagator can then be written as

$$P(x,t) \;=\; \begin{cases} P_{\rm sub}(x,t) \;\text{ for } t<t_{\rm d} \\[2mm] \int_{-\infty}^{\infty}{\rm d}x' P_{\rm comp}(x-x',t-t_{\rm d})P_{\rm sub}(x',t_{\rm d}) \\ \qquad\qquad \text{for } t>t_{\rm d}\end{cases} \tag{22}$$

Although this propagator permits diffusion on long time scales, it does not allow for any subdiffusion at all at $t>t_{\rm d}$. In reality, one would imagine that particles perform a sequence of subdiffusive episodes with a mean duration equal to $t_{\rm d}$, between which they decorrelate from the magnetic structures. In contrast to this, propagator (22) simply releases all particles from the correlated field after a time $t_{\rm d}$. We have also investigated an alternative picture in which particles are released from a correlated patch once they have diffused along the magnetic field for more than a "decorrelation length". This approach is closer to the physical picture of propagation in a stationary magnetic field, as discussed by Duffy et al. (1995). It leads to results which differ little from those described below.

Introducing dimensionless variables appropriate to the sub-diffusive case,

$$\xi=\frac{u^{1/3}}{D_{\rm M}^{2/3}\kappa_{\parallel}^{1/3}}x \qquad \tau=\frac{u^{4/3}}{D_{\rm M}^{2/3}\kappa_{\parallel}^{1/3}}t \tag{23}$$

enables the dimensionless sub-diffusive and diffusive propagators to be written

$$\begin{aligned} P_{\rm sub}(\xi,\tau) \;&=\; \frac{1}{2\pi}\int_0^{\infty}\frac{{\rm d}s}{\sqrt{s\tau}}\exp\left[-\xi^2/(4s)-s^2/(4\tau)\right] \\ P_{\rm comp}(\xi,\tau) \;&=\; \frac{1}{\sqrt{4\pi\hat\kappa_{\rm comp}\tau}}\exp\left[-\xi^2/(4\hat\kappa_{\rm comp}\tau)\right] \end{aligned} \tag{24}$$

Here $\hat\kappa_{\rm comp}=\kappa_{\rm comp}/(D_{\rm M}^{2/3}\kappa_{\parallel}^{1/3}u^{2/3})$ is the dimensionless form of the compound diffusion coefficient. This is closely related to the dimensionless decorrelation time $\tau_{\rm d}$. Evaluating the mean square deviation as a function of time, one finds

$$\left\langle\Delta\xi^2\right\rangle \;=\; \int_{-\infty}^{\infty}{\rm d}\xi\,\xi^2P(\xi,\tau) \tag{25}$$

$$\;=\; \begin{cases} 4\sqrt{\tau/\pi} & \text{for } \tau<\tau_{\rm d} \\[2mm] 4\sqrt{\tau_{\rm d}/\pi}+2\hat\kappa_{\rm comp}(\tau-\tau_{\rm d}) & \text{for } \tau>\tau_{\rm d}\end{cases} \tag{26}$$

Since the propagator itself is continuous at $\tau=\tau_{\rm d}$, so too is $\left\langle\Delta\xi^2\right\rangle$. Choosing $\hat\kappa_{\rm comp}=1/\sqrt{\pi\tau_{\rm d}}$ means that the first derivative is also continuous, and this is the value we adopt below. (It should be noted that the arguments leading, for example, to Eq. (10) determine $\hat\kappa_{\rm comp}$ only to within a factor of the order of unity.)

[1] We are grateful to Prof. Luke Drury for pointing out this result to us

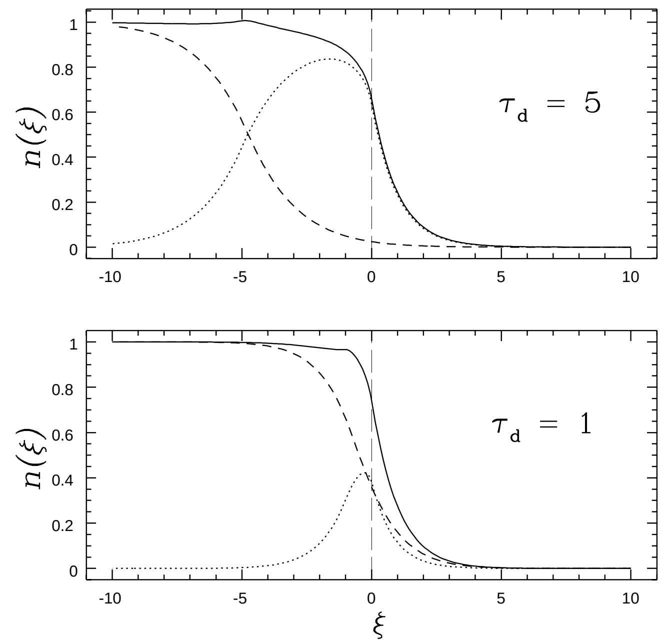

Fig. 1. The particle density as a function of distance from a moving boundary for two values of the dimensionless decorrelation time $\tau_{\rm d}$. The dotted line represents $n_1(\xi,\tau_{\rm d})$, i.e., those particles injected at the boundary within the last correlation time, which subsequently sub-diffused. The dashed line depicts $n_2(\xi,\tau_{\rm d})$, i.e., particles injected before this. After an initial sub-diffusive period, these performed standard diffusion. The total particle density is indicated by the solid line. The length scale is in dimensionless units (see Eq. 23) and the density is normalised to unity far downstream.

The stationary density at the shock for the mixed propagator Eq. (22) is made up of two parts:

$$n(\xi,\tau_{\rm d}) = \underbrace{\int_0^{\tau_{\rm d}} {\rm d}\tau P_{\rm sub}(\xi+\tau,\tau)}_{n_1(\xi,\tau_{\rm d})}$$
$$+ \underbrace{\int_{\tau_{\rm d}}^{\infty} {\rm d}\tau \int_{-\infty}^{\infty} {\rm d}\xi' P_{\rm comp}(\xi+\tau-\xi',\tau-\tau_{\rm d}) P_{\rm sub}(\xi',\tau_{\rm d})}_{n_2(\xi,\tau_{\rm d})} \quad (27)$$

The density $n_1(\xi,\tau_{\rm d})$ consists of those particles which have been injected less than one decorrelation time ago, and have consequently propagated by pure sub-diffusion. Particles injected before this contribute to $n_2(\xi,\tau_{\rm d})$, and have undergone a period of Markovian diffusion after decorrelating from their original field line. Evaluating these quantities involves, for n_1, the numerical computation of a double integral. On the other hand, the expression for n_2 may be reduced analytically to a single integral. Figure 1 shows these quantities for two values of the dimensionless decorrelation time. One feature apparent from this figure is the irregular behaviour at a distance $\xi \approx \tau_{\rm d}$ into the downstream medium. This is where, on average, our model propagator releases particles into diffusive propagation. The precise form of the density profile at this point is sensitive to the value chosen for $\hat{\kappa}_{\rm comp}$. Since a realistic propagator would not decorrelate particles all at the same time, and would also reimpose sub-diffusion after decorrelation, the feature displayed has no direct physical significance. Nevertheless, this model propagator enables one to get a qualitative picture of the density profile further upstream and downstream, and also indicates the length scale on which a more realistic profile would vary.

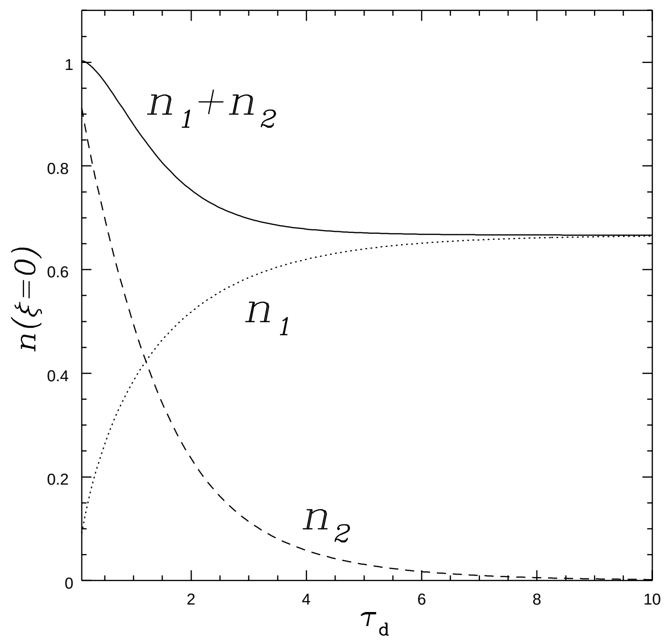

Fig. 2. The density of particles at the shock front (i.e., moving boundary) as a function of the decorrelation time. Shown are the recently injected, sub-diffusing particles n_1, the older, diffusing particles n_2 and the total density. As in Fig. 1, the density is normalised to take the value unity far downstream.

The power-law index of accelerated particles can be found from the values of n_1 and n_2 at the shock front using Eq. (18). In this case, n_1 also reduces to a single numerical integration. These quantities are plotted as a function of $\tau_{\rm d}$ in Fig. 2. The corresponding power law index of accelerated particles is shown in Fig. 3. Here it can be seen that the spectrum steepens rapidly as the decorrelation time increases. The index associated with sub-diffusion is achieved for decorrelation times greater than a few in dimensionless units, i.e.,

$$t_{\rm d} \gtrsim \frac{D_{\rm M}^{2/3}\kappa_{\parallel}^{1/3}}{u_2^{4/3}} \quad (28)$$

This quantity is approximately equal to the acceleration time found by Duffy et al. (1995). Thus, the steeper sub-diffusive spectrum applies when particles are able to increase their energy substantially before decorrelating from the magnetic field.

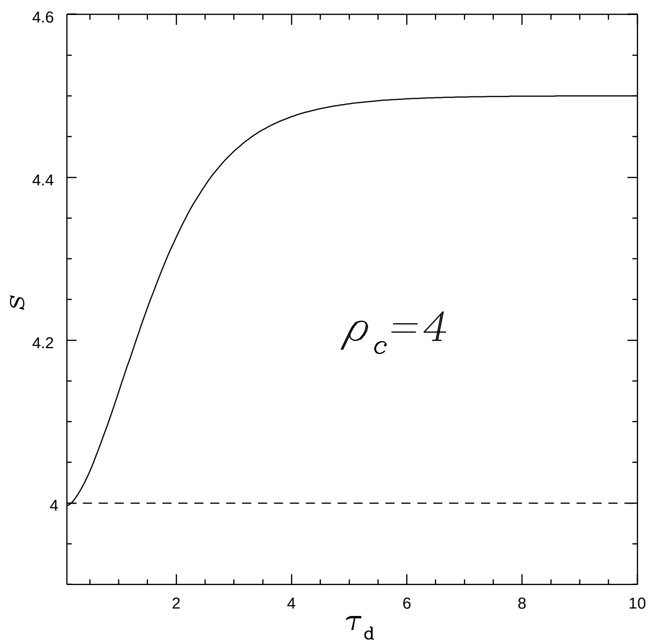

Fig. 3. The power law index of accelerated particles at a shock front of compression ratio 4, computed according to Eq. (18). It is assumed that after injection at the shock front particles propagate by sub-diffusion for times less than $\tau_{\rm d}$. Subsequently they diffuse. Only if $\tau_{\rm d}$ is short is the relatively flat spectrum of diffusive acceleration maintained. The time axis is in dimensionless units given in Eq. (23).

4. Discussion

The main conclusion of this paper is that stochastic test particle acceleration at a strong shock which is predominantly perpendicular does not necessarily produce the canonical $n(p) \propto p^{-2}$ spectrum expected from diffusive particle acceleration. Although much depends on the length scales and relative strengths of both the macroscopic inhomogeneities and the microscopic scattering, which, taken together, determine the decorrelation time $t_{\rm d}$, Eqs. (19) and (21) show that the spectrum produced depends in general on the properties of the turbulence, through the parameter β where $\langle \Delta x^2 \rangle \propto t^\beta$.

We have made several simplifying assumptions in order to arrive at an analytic treatment of this problem. As in the case of diffusive acceleration, we have assumed that the presence of the shock front does not have an important effect on the spatial transport of the particles, i.e., that the solution of the particle transport problem at a shock front can be obtained from a consideration of the case in which an imaginary boundary moves through a uniform stationary medium. This is certainly an idealisation. In a realistic situation the plasma upstream and downstream of a shock front probably supports turbulence of different character and amplitude. Furthermore, the plasma velocity and the magnetic field strength are discontinuous. Thus, we should allow for some change in the parameters of the propagator on crossing the shock, and perhaps also for a change of its functional form. Nevertheless, we feel that the essential physical feature introduced by the anomalous transport process – the steepening of the spectrum – is captured in our simple approach. This is because the nature of the transport in the upstream medium has very little effect on the particle spectrum, provided all particles entering this region are returned to the shock. The slope of the spectrum is determined by competition between energy gain on crossing the shock and escape downstream. In the case of sub-diffusion, a particle is always tied to the same field line. If we consider a particle at the shock front moving into the downstream medium, it is evident that its escape probability is determined by the average distance it can travel along its field line before encountering the shock again and does not depend on the nature of that portion of the field line which lies upstream. The situation is different if we are interested in the spatial dependence of the particle density upstream, or in the time taken to perform a cycle of crossing and recrossing, since then the nature of propagation in the upstream field is important.

Another of our assumptions – that the magnetic field lines themselves diffuse – is not readily lifted. Our calculation requires a specific model of the magnetic field, and we have too little knowledge of the properties of turbulence in the vicinity of shocks to do anything more realistic than simply assume diffusive behaviour. This corresponds to the case of sub-diffusion, $\beta = 1/2$. However, any situation in which the magnetic field plays a role in inhibiting particle transport is likely to resemble that of sub-diffusion, or, more generally, the case $\beta < 1$. In this connection, it is interesting to note that steep spectra have been found in numerical simulations of acceleration in random magnetic fields by Ballard & Heavens (1992). Although there is no well-defined average field direction in their computations, so that the shock cannot be described as quasi-perpendicular, the fact that particles diffuse freely in only one dimension, which does not always correspond with the direction of the shock normal, leads one to suspect that here too, anomalous transport may be an important factor. However, as well as the effects described above, their results may also be influenced by the relativistic flow speeds they assumed.

If particles are accelerated over several decades of momentum, as is thought to be the case in supernova remnants, it may be that different types of transport dominate in different ranges of p. One would expect that low energy

particles which have relatively small gyro-radii compared to the correlation length of the magnetic field might be more closely tied to the magnetic field lines than high energy ones, and so suffer more from sub-diffusive type effects. We have not presented a detailed discussion of this effect here, which may, nevertheless, be important in the problem of the acceleration of cosmic rays (Duffy et al. 1995).

Finally, the relative steepening of the spectral index for $\beta = 1/2$ also has important implications for the acceleration of electrons in astrophysical sources of synchrotron emission. In particular, a correlation should arise between the obliquity of the shock and the spectral index of the radiation. At perpendicular shocks, sub-diffusion can lead to particle spectra given by Eq. (19), which, even for uncooled electrons, would produce a relatively steep synchrotron spectrum of $F_\nu \propto \nu^{-0.75}$.

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